




CONSTRUCTION FOR q -HYPERGEOMETRIC BERNOULLI POLYNOMIALS OF A COMPLEX VARIABLE WITH APPLICATIONS COMPUTER MODELING

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Abstract. Two new extensions of the familiar Bernoulli polynomials are considered by using q -sine, q -cosine, q -hypergeometric and q -exponential functions. We call q -sine and q -cosine hypergeometric Bernoulli polynomials. Then, diverse formulas and properties for these polynomials, such as summation formulas, addition formulas, q -derivative properties, q -integral representations and some correlations are derived. Also, q -sine and q -cosine hypergeometric Bernoulli polynomials with two parameters are introduced and some relations and identities are investigated. Furthermore, some computational values are given by tables, and the beautiful zeros representations of the q -sine hypergeometric Bernoulli polynomials and q -cosine hypergeometric Bernoulli polynomials are showed by the figures.

Keywords: q -Bernoulli polynomials, q -hypergeometric Bernoulli numbers, q -numbers, q -difference equation.

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1 Introduction

The Appell polynomials $A_j(\psi)$, a particular case of Sheffer's polynomials, are defined by (Avram & Taqqu, 1987)

$$f(z)e^{\psi z} = \sum_{j=0}^{\infty} A_j(\psi) \frac{z^j}{j!}, \quad (1)$$

where f is a formal series in z , have possessed significant utilizations in several branches of theoretical physics, mathematics and chemistry (Alam et al., 2022; Al-Salam, 1959, 1976; Aoki et

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al., 2019; Corcino et al., 2022; Guan et al; 2023; Hu & Kim, 2018; Hassen & Nguyen, 2008; Khan et al., 2022, 2023a, 2023b; Khan, 2022a, 2022b; Norlund, 1924; Rao et al., 2023; Ryoo & Kang, 2020). One of the famous members of Appell polynomials is the Bernoulli polynomials $\mathbb{B}_j(\psi)$, generated by taking $f(z) = \frac{z}{e^z-1}$ in (1.1). Namely, these polynomials $\mathbb{B}_j(\psi)$ are introduced by

$$\frac{z}{e^z-1}e^{\psi z} = \sum_{j=0}^{\infty} B_j(\psi) \frac{z^j}{j!} \quad (|z| < 2\pi), \quad (2)$$

and also $B_j(0) := B_j$ are called Bernoulli numbers, (Aoki et al., 2019; Hu & Kim, 2018; Kamano, 2010; Khan, 2022b; Nguyen & Cheung, 2014). The mentioned polynomials and numbers and possess many useful applications in analytic number theory, numerical analysis, combinatorics and so on.

In 1924, Norlund (1924) considered the extended higher order Bernoulli polynomials as follows:

$$\left(\frac{z}{e^z-1}\right)^{\tau} e^{\psi z} = \sum_{j=0}^{\infty} B_j^{(\tau)}(\psi) \frac{z^j}{j!} \quad (|z| < 2\pi). \quad (3)$$

In the special case $\psi = 0$, $B_j^{(\tau)}(0) := B_j^{(\tau)}$ are termed the extended higher order Bernoulli numbers.

Howard (1967a, 1967b) gave the extensions of Bernoulli polynomials as follows

$$\frac{z^2 e^{\psi z}/2}{e^z-1-z} = \sum_{j=0}^{\infty} A_j^{(\tau)}(\psi) \frac{z^j}{j!}, \quad (4)$$

and in more general terms, for all natural numbers N

$$\frac{\frac{z^N}{N!}}{e^z - T_{N-1}(z)} e^{\psi z} = \sum_{j=0}^{\infty} \mathbb{B}_j(N, \psi) \frac{z^j}{j!}, \quad (5)$$

where

$$T_{N-1}(z) = \sum_{j=0}^{N-1} \frac{z^j}{j!}.$$

Upon setting $N = 1$ and $N = 2$, respectively, (5) becomes to (1) and (4). The mentioned polynomials $\mathbb{B}_j(N, \psi)$ are called hypergeometric Bernoulli polynomials, and also $\mathbb{B}_j(N, 0) := \mathbb{B}_j(N)$ are termed hypergeometric Bernoulli numbers.

For $r, N \in \mathbb{N}$, the hypergeometric Bernoulli polynomials $\mathbb{B}_j^{(r)}(N, \psi)$ of order r are introduced by (Aoki et al., 2019; Hu & Kim, 2018; Kamano, 2010; Khan, 2022b; Nguyen & Cheung, 2014)

$$\left(\frac{\frac{z^N}{N!}}{e^z - T_{N-1}(z)}\right)^r e^{\psi z} = \frac{1}{{}_1F_1(1; N+1; z)^r} e^{\psi z} = \sum_{j=0}^{\infty} \mathbb{B}_j^{(r)}(N, \psi) \frac{z^j}{j!}. \quad (6)$$

The hypergeometric Bernoulli numbers of order r are introduced by taking $\psi = 0$ in (6), namely $\mathbb{B}_j^{(r)}(N, 0) := \mathbb{B}_j^{(r)}(N)$ (Kamano, 2010; Khan, 2022b; Nguyen and Cheung, 2014). Note that

$$\mathbb{B}_j^{(1)}(N, \psi) := \mathbb{B}_j(N, \psi) \text{ and } \mathbb{B}_j^{(1)}(N, 0) := \mathbb{B}_j(N).$$

Inspired by the significance and potential utilizations, Bernoulli polynomials with multifarious extensions have been studied by many authors (Al-Salam, 1959, 1976; Guan et al., 2023; Kang and Ryoo, 2020; Mbutngam and Sadjang, 2021; Muhiuddin et al., 2021). Recently, Kang & Ryoo (2020) and Ryoo & Kang (2020) considered the q -Bernoulli polynomials of a complex

variable, by which the polynomials defined the parametric types of the aforesaid polynomials, by separating the real and imaginary parts. In the present article, we first provide some preliminary definitions and results useful for the next sections. In Section 3, we define two q -analogs of the hypergeometric Bernoulli polynomials of complex variable and acquire a lot of formulas and identities of these polynomials. In Section 4, we derive q -hypergeometric Bernoulli polynomials of complex variable with two parameters M and N and establish some beautiful results. In Section 5, we derive some numerical values of q -extensions of the hypergeometric Bernoulli polynomials of complex variable and draw some beautiful graphs. Then, in the last section, we analyze the results derived from this study.

2 Preliminary Information

Here, the notations and definitions of the theory of q -calculus are given (Kac& Cheung, 2002; Koekoek et al., 2010; Rao et al., 2023) for more detailed information about q -calculus.

The q -Pochhammer numbers $(\tau; q)_j$ are given by

$$(\tau; q)_0 = 1 \text{ and } (\tau; q)_j = \prod_{m=0}^{j-1} (1 - q^m \tau) \text{ for } j \in \mathbb{N}.$$

For $j = \infty$, we note that

$$(\tau; q)_\infty = \prod_{m=0}^{\infty} (1 - q^m \tau), \quad |q| < 1$$

and for $j \in \mathbb{N}$, we give

$$(\tau; q)_j = \frac{(\tau; q)_\infty}{(\tau q^j; q)_\infty}.$$

The q -extension of a number τ and q -extension of the factorial numbers $j!$ are provided by

$$[\tau]_q = \frac{1 - q^\tau}{1 - q} \text{ for } q \in \mathbb{C} - \{1\},$$

$$[j]_q! = \frac{(q; q)_j}{(1 - q)^j} = [1]_q [2]_q \cdots [j]_q \text{ for } q \neq 1; j \in \mathbb{N} \text{ with } [0]_q! := 1.$$

The q -extension of the binomial coefficient is provided by

$$\binom{j}{k}_q = \frac{(q; q)_j}{(q; q)_k (q; q)_{j-k}} = \frac{[j]_q!}{[k]_q! [j-k]_q!}, \quad k = 0, 1, \dots, j.$$

The two q -power basis are defined by

$$(\mu \oplus \nu)_q^j := (\mu + \nu)(\mu + q\nu) \cdots (\mu + q^{j-1}\nu)$$

and

$$(\mu \ominus \nu)_q^j := (\mu - \nu)(\mu - q\nu) \cdots (\mu - q^{j-1}\nu),$$

which provides

$$(\mu \oplus \nu)_q^j = (\mu \ominus (-\nu))_q^j = \sum_{k=0}^j \binom{j}{k}_q q^{k(k-1)/2} \mu^{j-k} \nu^k \text{ for } j \in \mathbb{N}_0. \quad (7)$$

The two q -extensions of the exponential function e^μ are provided by

$$e_q(\mu) = \frac{1}{((1 - q)\mu; q)_\infty} = \sum_{j=0}^{\infty} \frac{\mu^j}{[j]_q!}, \quad 0 < |q| < 1 \text{ and } |1 - q|^{-1} > |\mu| \quad (8)$$

and

$$E_q(\mu) = (-(1-q)\mu; q)_\infty = \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}}{[j]_q!} \mu^j \text{ for } 0 < |q| < 1 \text{ and } \mu \in \mathbb{C}, \quad (9)$$

which hold the following equations (Kac & Cheung, 2002)

$$1 = e_q(\mu)E_q(-\mu) = e_q(-\sigma)E_q(\sigma).$$

Definition 1. Let $\psi, \nu \in \mathbb{C}$ and $j \in \mathbb{N}_0$. The q -addition is introduced by the following summation (Kac & Cheung, 2002; Koekoek et al., 2010)

$$(\psi \oplus_q \nu)^j := \sum_k^j \psi^k \nu^{j-k} \binom{j}{k}_q. \quad (10)$$

Definition 2. The q -hypergeometric function ${}_r\phi_s$ is introduced by the following infinite series (Kac & Cheung, 2002; Koekoek et al., 2010)

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) = \sum_{k=0}^{\infty} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \frac{z^k}{(q; q)_k}, \quad (11)$$

where the notation $(a_1, \dots, a_r; q)_k$ equals to $(a_1; q)_k \cdots (a_r; q)_k$. It is worth noting that:

$${}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix} \middle| q; z \right) = \sum_{j=0}^{\infty} \frac{(a; q)_j}{(q; q)_j} z^j = \frac{(az; q)_\infty}{(z; q)_\infty} \text{ for } |z| < 1 \text{ and } 0 < |q| < 1. \quad (12)$$

The q -derivative operator is given by (Kac & Cheung, 2002)

$$D_{q,z}f(z) = \frac{f(qz) - f(z)}{qz - z} \text{ for } 0 < |q| < 1,$$

and $D_{q,z}f(0) = f'(0)$ provided that f is differentiable at $x = 0$.

For $a, b \in \mathbb{R}$ with $0 < a < b$, the q -extension of definite integral is provided by (Kac & Cheung, 2002)

$$\int_0^b f(\psi) d_q \psi = (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b) \quad (13)$$

with

$$\int_a^b f(\psi) d_q \psi = \int_0^b f(\psi) d_q \psi - \int_0^a f(\psi) d_q \psi. \quad (14)$$

The q -extension of Gamma function is provided by (Mboutngam & Sadjang, 2021)

$$\Gamma_q(\psi) = \frac{(q; q)_\infty}{(q^\psi; q)_\infty} (1-q)^{1-\psi} \text{ with } 0 < q < 1, \quad (15)$$

which means the following equalities

$$\Gamma_q(\psi) = (1-q)^{1-\psi} (q; q)_{\psi-1}$$

and

$$\Gamma_q(\psi + 1) = [\psi]_q \Gamma_q(\psi) \text{ with } \Gamma_q(1) = 1.$$

For $1 > |q| > 0$ and $\beta, \tau \in \mathbb{C}$, we get (Mboutngam & Sadjang, 2021)

$$\binom{\tau}{k}_q = (-1)^k q^{\tau k - \binom{k}{2}} \frac{(q^{-\tau}; q)_k}{(q; q)_k}$$

and

$$\binom{\tau}{\beta}_q = \frac{\Gamma_q(\tau + 1)}{\Gamma_q(\tau - \beta + 1) \Gamma_q(\beta + 1)} = \frac{(q^{\tau-\beta+1}; q)_\infty (q^{\beta+1}; q)_\infty}{(q^{\tau+1}; q)_\infty (q; q)_\infty}.$$

Definition 3. Let $s, t > 0$. The q -extension of Beta function is introduced by (Kac & Cheung, 2002; Mboutngam & Sadjang, 2021)

$$B_q(z, s) = \int_0^1 \psi^{z-1} (1 \ominus q\psi)_q^{s-1} d_q \psi. \tag{16}$$

It is noticed that the q -Gamma function and the q -Beta function satisfy the following relation (Kac & Cheung, 2002; Koekoek et al., 2010)

$$B_q(z, s) = \frac{\Gamma_q(z)\Gamma_q(s)}{\Gamma_q(z+s)}.$$

The q -extensions of Bernoulli and Euler polynomials are defined by (Alam et al., 2022; Al-Salam, 1959, 1976; Kac & Cheung, 2002; Koekoek et al., 2010; Kang & Ryoo, 2020; Ryoo & Kang, 2020)

$$\sum_{j=0}^{\infty} \mathbb{B}_{j,q}(\psi) \frac{z^j}{[j]_q!} = \frac{ze_q(\psi z)}{e_q(z) - 1} \text{ and } \sum_{j=0}^{\infty} \mathbb{E}_{j,q}(\psi) \frac{z^j}{[j]_q!} = \frac{2}{e_q(z) + 1} e_q(\psi z),$$

respectively. The cosine and sine forms of the q -Bernoulli and q -Euler polynomials are considered by Kang and Ryoo (Kang & Ryoo, 2020; Ryoo & Kang, 2020) as follows:

$$\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(C)}(\psi, \nu) \frac{z^j}{[j]_q!} = \frac{z}{e_q(z) - 1} e_q(\psi z) \cos_q(\nu z) = \sum_{j=0}^{\infty} \frac{\mathbb{B}_{j,q}((\psi \oplus i\nu)_q) + \mathbb{B}_{j,q}((\psi \ominus i\nu)_q)}{2} \frac{z^j}{[j]_q!}, \tag{17}$$

$$\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(S)}(\psi, \nu) \frac{z^j}{[j]_q!} = \frac{z}{e_q(z) - 1} e_q(\psi z) \sin_q(\nu z) = \sum_{j=0}^{\infty} \frac{\mathbb{B}_{j,q}((\psi \oplus i\nu)_q) - \mathbb{B}_{j,q}((\psi \ominus i\nu)_q)}{2i} \frac{z^j}{[j]_q!}, \tag{18}$$

and

$$\sum_{j=0}^{\infty} \mathbb{E}_{j,q}^{(C)}(\psi, \nu) \frac{z^j}{[j]_q!} = \frac{2}{e_q(z) + 1} e_q(\psi z) \cos_q(\nu z) = \sum_{j=0}^{\infty} \frac{\mathbb{E}_{j,q}((\psi \oplus i\nu)_q) + \mathbb{E}_{j,q}((\psi \ominus i\nu)_q)}{2} \frac{z^j}{[j]_q!}, \tag{19}$$

$$\sum_{j=0}^{\infty} \mathbb{E}_{j,q}^{(S)}(\psi, \nu) \frac{z^j}{[j]_q!} = \frac{2}{e_q(z) + 1} e_q(\psi z) \sin_q(\nu z) = \sum_{j=0}^{\infty} \frac{\mathbb{E}_{j,q}((\psi \oplus i\nu)_q) - \mathbb{E}_{j,q}((\psi \ominus i\nu)_q)}{2i} \frac{z^j}{[j]_q!}, \tag{20}$$

where we have used the q -cosine and q -sine polynomials given by (Kang & Ryoo, 2020; Mboutngam & Sadjang, 2021; Ryoo & Kang, 2020):

$$e_q(\psi z) \cos_q(\nu z) = \sum_{r=0}^{\infty} C_{r,q}(\psi, \nu) \frac{z^r}{[r]_q!} \tag{21}$$

and

$$e_q(\psi z) \sin_q(\nu z) = \sum_{r=0}^{\infty} S_{r,q}(\psi, \nu) \frac{z^r}{[r]_q!}, \tag{22}$$

where

$$\frac{e_q(iz) + e_q(-iz)}{2} = \cos_q(z) \text{ and } \frac{e_q(iz) - e_q(-iz)}{2i} = \sin_q(z).$$

Note that

$$C_{r,q}(\psi, \nu) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^j \binom{r}{2j}_q (-1)^j q^{2j-1} \psi^{r-2j} \nu^{2j} \quad (23)$$

and

$$S_{r,q}(\psi, \nu) = \sum_{j=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r}{2j+1}_q (-1)^j q^{(2j+1)j} \psi^{r-2j-1} \nu^{2j+1}. \quad (24)$$

The q -hypergeometric Bernoulli polynomials are given by (Mboutngam & Sadjang, 2021)

$$\frac{z^N}{[N]_q!} e_q(\psi z) = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}(N, \psi) \frac{z^j}{[j]_q!}, \quad (25)$$

where

$$T_{N,q}(z) = \sum_{j=0}^N \frac{z^j}{[j]_q!}.$$

Also when $\psi = 0$, $\mathbb{B}_{j,q}(N, 0) := \mathbb{B}_{j,q}(N)$ are called the q -hypergeometric Bernoulli numbers.

3 On q -Hypergeometric Bernoulli Polynomials of Complex Variable

Here, we define q -sine and q -cosine hypergeometric Bernoulli polynomials and investigate some of their properties and relations. First, we consider

$$\frac{z^N}{[N]_q!} e_q((\psi + i\nu)z) = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}(N, \psi + i\nu) \frac{z^j}{[j]_q!}. \quad (26)$$

By using the following property

$$e_q((\psi + i\nu)z) = e_q(\psi z) (\cos_q \nu z + i \sin_q \nu z) \quad (27)$$

and by (26), we observe that

$$\sum_{j=0}^{\infty} \mathbb{B}_{j,q}(N, \psi + i\nu) \frac{z^j}{[j]_q!} = \frac{z^N}{[N]_q!} e_q(\psi z) (\cos_q \nu z + i \sin_q \nu z), \quad (28)$$

and

$$\sum_{j=0}^{\infty} \mathbb{B}_{j,q}(N, \psi - i\nu) \frac{z^j}{[j]_q!} = \frac{z^N}{[N]_q!} e_q(\psi z) (\cos_q \nu z - i \sin_q \nu z). \quad (29)$$

By means of (28) and (29), we get

$$\frac{z^N}{[N]_q!} e_q(\psi z) \cos_q(\nu z) = \sum_{j=0}^{\infty} \left(\frac{\mathbb{B}_{j,q}(N, \psi + i\nu) + \mathbb{B}_{j,q}(N, \psi - i\nu)}{2} \right) \frac{z^j}{[j]_q!}, \quad (30)$$

and

$$\frac{z^N}{[N]_q!} e_q(\psi z) \sin_q(\nu z) = \sum_{j=0}^{\infty} \left(\frac{\mathbb{B}_{j,q}(N, \psi + i\nu) - \mathbb{B}_{j,q}(N, \psi - i\nu)}{2i} \right) \frac{z^j}{[j]_q!}. \quad (31)$$

Inspired by (30) and (31), we provide the following definition.

Definition 4. Let $j \geq 0$. We define q -cosine and q -sine hypergeometric Bernoulli polynomials, $\mathbb{B}_{j,q}^{(c)}(N, \psi, \nu)$ and $\mathbb{B}_{j,q}^{(s)}(N, \psi, \nu)$ as follows:

$$\frac{z^N}{[N]_q!} e_q(\psi z) \cos_q(\nu z) = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!}, \quad (32)$$

and

$$\frac{z^N}{[N]_q!} e_q(\psi z) \sin_q(\nu z) = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(s)}(N, \psi, \nu) \frac{z^j}{[j]_q!}, \quad (33)$$

respectively.

We note that

$$\mathbb{B}_{j,q}^{(c)}(N, 0, 0) := \mathbb{B}_{j,q}(N) \quad \text{and} \quad \mathbb{B}_{j,q}^{(s)}(N, 0, 0) := 0 \quad \text{for } j \geq 0.$$

From (30)-(33), we have

$$\frac{\mathbb{B}_{j,q}(N, \psi + i\nu) + \mathbb{B}_{j,q}(N, \psi - i\nu)}{2} = \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu), \quad (34)$$

$$\frac{\mathbb{B}_{j,q}(N, \psi + i\nu) - \mathbb{B}_{j,q}(N, \psi - i\nu)}{2i} = \mathbb{B}_{j,q}^{(s)}(N, \psi, \nu). \quad (35)$$

Remark 1. For $\psi = 0$ in (32) and (33), we obtain

$$\frac{z^N}{[N]_q!} \cos_q(\nu z) = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \nu) \frac{z^j}{[j]_q!}, \quad (36)$$

and

$$\frac{z^N}{[N]_q!} \sin_q(\nu z) = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(s)}(N, \nu) \frac{z^j}{[j]_q!}, \quad (37)$$

respectively.

Now, we begin with obtaining some relations and properties of the aforementioned polynomials.

Theorem 1. Let $j \geq 0$. We have

$$\mathbb{B}_{j,q}^{(c)}(N, \nu) = \sum_{v=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j+v}{2v}_q q^{(2v-1)v} (-1)^v \nu^{2v} \mathbb{B}_{j-2v,q}(N), \quad (38)$$

and

$$\mathbb{B}_{j,q}^{(s)}(N, \nu) = \sum_{v=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j+v}{2v+1}_q (-1)^v q^{(2v+1)v} \nu^{2v+1} \mathbb{B}_{j-2v-1,q}(N). \quad (39)$$

Proof. By (36) and (37), we attain

$$\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \nu) \frac{z^j}{[j]_q!} = \frac{z^N}{[N]_q!} \cos_q(\nu z)$$

$$\begin{aligned}
&= \sum_{j=0}^{\infty} \mathbb{B}_{j,q}(N) \frac{z^j}{[j]_q!} \sum_{v=0}^{\infty} q^{(2v-1)v} (-1)^v \nu^{2v} \frac{z^v}{[2v]_q!} \\
&= \sum_{j=0}^{\infty} \left(\sum_{v=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j+v}{2v}_q (-1)^v q^{(2v-1)v} \nu^{2v} \mathbb{B}_{j-2v,q}(N) \right) \frac{z^j}{[j]_q!}, \tag{40}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(s)}(N, \nu) \frac{z^j}{[j]_q!} = \frac{\frac{z^N}{[N]_q!}}{e_q(z) - T_{N-1,q}(z)} \sin_q(\nu z) \\
&= \sum_{j=0}^{\infty} \left(\sum_{v=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j}{2v+1}_q (-1)^v q^{(2v+1)v} \nu^{2v+1} \mathbb{B}_{j-2v-1,q}(N) \right) \frac{z^j}{[j]_q!}. \tag{41}
\end{aligned}$$

Therefore, by (40) and (41), we get (38) and (39). \square

Theorem 2. *Let $j \geq 0$. We have*

$$\begin{aligned}
\mathbb{B}_{j,q}(N, \psi + i\nu) &= \sum_{u=0}^j \binom{j}{u}_q (\psi + i\nu)^u \mathbb{B}_{j-u,q}(N) \\
&= \sum_{u=0}^j \binom{j}{u}_q (i\nu)^u \mathbb{B}_{j-u,q}(N, \psi), \tag{42}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{B}_{j,q}(N, \psi - i\nu) &= \sum_{u=0}^j \binom{j}{u}_q (\psi - i\nu)^u \mathbb{B}_{j-u,q}(N) \\
&= \sum_{u=0}^j \binom{j}{u}_q (-1)^u (i\nu)^u \mathbb{B}_{j-u,q}(N, \psi). \tag{43}
\end{aligned}$$

Proof. By using (28) and (29), we obtain (38) and (39). So we omit the details. \square

Theorem 3. *Let $j \geq 0$. We have*

$$\mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) = \sum_{v=0}^j \binom{j}{v}_q \mathbb{B}_{v,q}(N) C_{j-v,q}(\psi, \nu), \tag{44}$$

and

$$\mathbb{B}_{j,q}^{(s)}(N, \psi, \nu) = \sum_{v=0}^j \binom{j}{v}_q \mathbb{B}_{v,q}(N) S_{j-v,q}(\psi, \nu). \tag{45}$$

Proof. By Definition 4, we see that

$$\begin{aligned}
&\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!} = \frac{\frac{z^N}{[N]_q!}}{e_q(z) - T_{N-1,q}(z)} e_q(\psi z) \cos_q(\nu z) \\
&= \left(\sum_{j=0}^{\infty} C_{j,q}(\psi, \nu) \frac{z^j}{[j]_q!} \right) \left(\sum_{v=0}^{\infty} \mathbb{B}_{v,q}(N) \frac{z^v}{[v]_q!} \right) \\
&= \sum_{j=0}^{\infty} \left(\sum_{v=0}^j \binom{j}{v}_q \mathbb{B}_{v,q}(N) C_{j-v,q}(\psi, \nu) \right) \frac{z^j}{[j]_q!},
\end{aligned}$$

which means (44). The other proof is similar. So we omit it. \square

Theorem 4. Let $j \geq 0$. We have

$$C_{j,q}(\psi, \nu) = \sum_{k=0}^j \frac{\binom{j+N}{k}_q}{\binom{j+N}{j}_q} \mathbb{B}_{k,q}^{(c)}(N, \psi, \nu), \quad (46)$$

and

$$S_{j,q}(\psi, \nu) = \sum_{k=0}^j \mathbb{B}_{k,q}^{(s)}(N, \psi, \nu) \frac{\binom{j+N}{k}_q}{\binom{j+N}{j}_q}. \quad (47)$$

Proof. By Definition 4, we observe that

$$\begin{aligned} e_q(\psi z) \cos_q(\nu z) &= \frac{e_q(z) - T_{N-1,q}(z)}{[N]_q!} \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!} \\ &= [N]_q! \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!} \right) \left(\sum_{j=N}^{\infty} \frac{z^j}{[j]_q!} \right) \\ &= [N]_q! \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \mathbb{B}_{k,q}^{(c)}(N, \psi, \nu) \binom{j+N}{k}_q \right) \frac{z^j}{[j+N]_q!}. \end{aligned} \quad (48)$$

By (32) and (48), we get (46). Similarly, we can prove (2.22). So we omit the details. \square

Theorem 5. For $j \geq 0$, we have

$$\mathbb{B}_{j,q}^{(c)}(N, \psi + s, \nu) = \sum_{u=0}^j \binom{j}{u}_q \mathbb{B}_{u,q}^{(c)}(N, \psi, \nu) s^{j-u}, \quad (49)$$

and

$$\mathbb{B}_{j,q}^{(s)}(N, \psi + s, \nu) = \sum_{u=0}^j \binom{j}{u}_q \mathbb{B}_{u,q}^{(s)}(N, \psi, \nu) s^{j-u}. \quad (50)$$

Proof. By substituting ψ by $\psi + s$ in (32), we see that

$$\begin{aligned} \sum_{j=0}^{\infty} \mathbb{B}_j^{(c)}(N, \psi + s, \nu) \frac{z^j}{[j]_q!} &= \frac{\frac{z^N}{[N]_q!}}{e_q(z) - T_{N-1,q}(z)} e_q(\psi z) \cos_q(\nu z) e_q(sz) \\ &= \left(\sum_{j=0}^{\infty} \mathbb{B}_j^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!} \right) \left(\sum_{u=0}^{\infty} s^u \frac{z^u}{[u]_q!} \right) \\ &= \sum_{j=0}^{\infty} \left(\sum_{u=0}^j \binom{j}{u}_q \mathbb{B}_{u,q}^{(c)}(N, \psi, \nu) s^{j-u} \right) \frac{z^j}{[j]_q!}, \end{aligned}$$

which means the result in (49). The result (50) can be done similarly. So we omit the details. \square

Theorem 6. Let $j \geq 0$. We have

$$D_{q,\psi} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) = [j]_q \mathbb{B}_{j-1,q}^{(c)}(N, \psi, \nu), \quad (51)$$

$$D_{q,\nu} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) = -[j]_q \mathbb{B}_{j-1,q}^{(s)}(N, \psi, q\nu), \quad (52)$$

and

$$D_{q,\psi} \mathbb{B}_{j,q}^{(s)}(N, \psi, \nu) = [j]_q \mathbb{B}_{j-1,q}^{(s)}(N, \psi, \nu), \quad (53)$$

$$D_{q,\nu} \mathbb{B}_{j,q}^{(s)}(N, \psi, \nu) = [j]_q \mathbb{B}_{j-1,q}^{(c)}(N, \psi, q\nu). \quad (54)$$

Proof. Based on the following q -derivative properties (see Kac et al., 2001)

$$D_{q,\psi} e_q(\psi z) = z e_q(\psi z), \quad D_{q,\nu} \cos_q(\nu z) = -\sin_q(q\nu z) \quad \text{and} \quad D_{q,\nu} \sin_q(\nu z) = \cos_q(q\nu z)$$

and using (32), we compute

$$\begin{aligned} \sum_{j=1}^{\infty} D_{q,\psi} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!} &= \frac{z^N}{[N]_q!} D_{q,\psi} e_q(\psi z) \cos_q(\nu z) = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) \frac{z^{j+1}}{[j]_q!} \\ &= \sum_{j=1}^{\infty} \mathbb{B}_{j-1}^{(c)}(N, \psi, \nu) \frac{z^j}{[(j-1)]_q!} = \sum_{j=1}^{\infty} [j]_q \mathbb{B}_{j-1,q}^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!}, \end{aligned}$$

proving (51). Other (52), (53) and (54) can be derived similarly. So we omit the proof of (54). \square

Theorem 7. *Let $N \in \mathbb{N}$. We have*

$$\int_0^1 (1 \ominus q\psi)_q^{N-1} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) d_q\psi = [N-1]_q! \sum_{k=0}^n \binom{j}{k}_q \frac{[j-k]_q!}{[N+j-k]_q!} \mathbb{B}_{k,q}^{(c)}(N, \nu), \quad (55)$$

and

$$\int_0^1 (1 \ominus q\psi)_q^{N-1} \mathbb{B}_{j,q}^{(s)}(N, \psi, \nu) d_q\psi = [N-1]_q! \sum_{k=0}^n \binom{j}{k}_q \frac{[j-k]_q!}{[N+j-k]_q!} \mathbb{B}_{k,q}^{(s)}(N, \nu). \quad (56)$$

Proof. From (32), we have

$$\Upsilon = \int_0^1 (1 \ominus q\psi)_q^{N-1} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) d_q\psi = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{k,q}^{(c)}(N, \nu) \int_0^1 (1 \ominus q\psi)_q^{N-1} \psi^{j-k} d_q\psi. \quad (57)$$

Using the definition of the q -Beta function given by (16) and (30), we get

$$\begin{aligned} \Upsilon &= \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{k,q}^{(c)}(N, \nu) B_q(N, j-k+1) \\ &= \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{k,q}^{(c)}(N, \nu) \frac{[N-1]_q! [j-k]_q!}{[N+j-k]_q!}. \end{aligned} \quad (58)$$

From (57) and (58), it follows that

$$\int_0^1 (1 \ominus q\psi)_q^{N-1} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) d_q\psi = [N-1]_q! \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{k,q}^{(c)}(N, \nu) \frac{[j-k]_q!}{[N+j-k]_q!}.$$

The complete proof of (55). The proof of (56) is similar. \square

Theorem 8. *The following relations hold:*

$$\begin{aligned} \mathbb{B}_{j,q}^{(c)}(N, (\psi \oplus r)_q, \nu) &= \sum_{l=0}^j \binom{j}{l} \mathbb{B}_{j-l,q}^{(c)}(N, \psi, \nu) q^{\binom{l}{2}} r^l, \\ \mathbb{B}_{j,q}^{(c)}(N, (\psi \ominus r)_q, \nu) &= \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(c)}(N, \psi, \nu) q^{\binom{l}{2}} (-1)^l r^k, \end{aligned}$$

and

$$\begin{aligned}\mathbb{B}_{j,q}^{(s)}(N, (\psi \oplus r)_q, \nu) &= \sum_{l=0}^j \binom{j}{l} \mathbb{B}_{j-l,q}^{(s)}(N, \psi, \nu) q^{\binom{l}{2}} r^l, \\ \mathbb{B}_{j,q}^{(s)}(N, (\psi \ominus r)_q, \nu) &= \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(s)}(N, \psi, \nu) q^{\binom{l}{2}} (-1)^l r^k.\end{aligned}$$

Proof. By Definition 4, we have

$$\begin{aligned}\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, (\psi \oplus r)_q, \nu) \frac{z^j}{[j]_q!} &= \frac{z^N}{[N]_q!} e_q(z) - T_{N-1,q}(z) e_q(z(\psi \oplus r)_q) \cos_q(\nu z) \\ &= \frac{z^N}{[N]_q!} e_q(z\psi) \cos_q(\nu z) E_q(zr) \\ &= \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu) \frac{z^j}{[j]_q!} \sum_{l=0}^{\infty} q^{\binom{l}{2}} r^l \frac{z^l}{[l]_q!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{l=0}^j \binom{j}{l} \mathbb{B}_{j-l,q}^{(c)}(N, \psi, \nu) q^{\binom{l}{2}} r^l \right) \frac{z^j}{[j]_q!}.\end{aligned}\tag{59}$$

Similar to (59), we observe that

$$\begin{aligned}\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(s)}(N, (\psi \ominus r)_q, \nu) \frac{z^j}{[j]_q!} &= \frac{z^N}{[N]_q!} e_q(z) - T_{N-1,q}(z) e_q(z(\psi \ominus r)_q) \sin_q(\nu z) \\ &= \frac{z^N}{[N]_q!} e_q(z\psi) \sin_q(\nu z) E_q(-zr) \\ &= \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(s)}(N, \psi, \nu) \frac{z^j}{[j]_q!} \sum_{l=0}^{\infty} q^{\binom{l}{2}} (-1)^l r^l \frac{z^l}{[l]_q!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(s)}(N, \psi, \nu) q^{\binom{l}{2}} (-1)^l r^k \right) \frac{z^j}{[j]_q!}.\end{aligned}\tag{60}$$

By (59) and (60), we can easily obtain the first and the third asserted results in the theorem. The others can be done similarly. So we omit them. \square

Now, we provide some corollaries of Theorem 8 as follows.

Corollary 1. *Let $j \geq 0$. By Theorem 8, we readily get*

$$\begin{aligned}&\mathbb{B}_{j,q}^{(c)}(N, (\psi \oplus r)_q, \nu) + \mathbb{B}_{j,q}^{(s)}(N, (\psi \ominus r)_q, \nu) \\ &= \sum_{k=0}^j \binom{j}{k}_q q^{\binom{j-k}{2}} r^{j-k} \left(\mathbb{B}_{k,q}^{(c)}(N, \psi, \nu) + (-1)^{j-k} \mathbb{B}_{k,q}^{(s)}(N, \psi, \nu) \right),\end{aligned}\tag{61}$$

and

$$\begin{aligned}&\mathbb{B}_{j,q}^{(s)}(N, (\psi \oplus r)_q, \nu) + \mathbb{B}_{j,q}^{(c)}(N, (\psi \ominus r)_q, \nu) \\ &= \sum_{k=0}^j \binom{j}{k}_q q^{\binom{j-k}{2}} r^{j-k} \left(\mathbb{B}_{k,q}^{(s)}(N, \psi, \nu) + (-1)^{j-k} \mathbb{B}_{k,q}^{(c)}(N, \psi, \nu) \right).\end{aligned}\tag{62}$$

Corollary 2. *Let $j \geq 0$. Then*

$$\begin{aligned} & \mathbb{B}_{j,q}^{(c)}(N, (\psi \oplus r)_q, \nu) + \mathbb{B}_{j,q}^{(c)}(N, (\psi \ominus r)_q, \nu) \\ &= \sum_{k=0}^j \binom{j}{k}_q q^{\binom{k}{2}} r^k \left(\mathbb{B}_{j-k,q}^{(c)}(N, \psi, \nu) + (-1)^k \mathbb{B}_{j-k,q}^{(c)}(N, \psi, \nu) \right), \end{aligned} \quad (63)$$

and

$$\begin{aligned} & \mathbb{B}_{j,q}^{(s)}(N, (\psi \oplus r)_q, \nu) + \mathbb{B}_{j,q}^{(s)}(N, (\psi \ominus r)_q, \nu) \\ &= \sum_{k=0}^j \binom{j}{k}_q q^{\binom{k}{2}} r^k \left(\mathbb{B}_{j-k,q}^{(s)}(N, \psi, \nu) + (-1)^k \mathbb{B}_{j-k,q}^{(s)}(N, \psi, \nu) \right). \end{aligned} \quad (64)$$

Corollary 3. *For $r = 1$ in Theorem 8, we have*

$$\begin{aligned} & \mathbb{B}_{j,q}^{(c)}(N, (\psi \oplus 1)_q, \nu) + \mathbb{B}_{j,q}^{(s)}(N, (\psi \ominus 1)_q, \nu) \\ &= \sum_{k=0}^j \binom{j}{k}_q q^{\binom{k}{2}} \left(\mathbb{B}_{j-k,q}^{(c)}(N, \psi, \nu) + (-1)^k \mathbb{B}_{j-k,q}^{(s)}(N, \psi, \nu) \right), \end{aligned} \quad (65)$$

and

$$\begin{aligned} & \mathbb{B}_{j,q}^{(s)}(N, (\psi \oplus 1)_q, \nu) + \mathbb{B}_{j,q}^{(c)}(N, (\psi \ominus 1)_q, \nu) \\ &= \sum_{k=0}^j \binom{j}{k}_q q^{\binom{j-k}{2}} \left(\mathbb{B}_{k,q}^{(s)}(N, \psi, \nu) + (-1)^{j-k} \mathbb{B}_{k,q}^{(c)}(N, \psi, \nu) \right). \end{aligned} \quad (66)$$

4 The q -Hypergeometric Bernoulli Polynomials of Complex Variable with Two Parameters

Here, we define q -cosine and q -sine hypergeometric Bernoulli polynomials with two parameters M and N and derive some of their relations and formulas. We first consider the following definitions:

$$\frac{e_q(\psi z) \cos_q(\nu z)}{{}_2\phi_1 \left(\begin{matrix} q^{M+1}, & 0 \\ & q^{M+N+1} \end{matrix} \middle| q(1-q)z \right)} = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!}, \quad (67)$$

and

$$\frac{e_q(\psi z) \sin_q(\nu z)}{{}_2\phi_1 \left(\begin{matrix} q^{M+1}, & 0 \\ & q^{M+N+1} \end{matrix} \middle| q(1-q)z \right)} = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(s)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!}. \quad (68)$$

Obviously, for $M = 0$ and N is a positive integer, we have $\mathbb{B}_{j,q}^{(c)}(0, N, \psi, \nu) := \mathbb{B}_{j,q}^{(c)}(N, \psi, \nu)$ and $\mathbb{B}_{j,q}^{(s)}(0, N, \psi, \nu) := \mathbb{B}_{j,q}^{(s)}(N, \psi, \nu)$, where $\mathbb{B}_{j,q}^{(c)}(N, \psi, \nu)$ and $\mathbb{B}_{j,q}^{(s)}(N, \psi, \nu)$ are called the q -cosine hypergeometric Bernoulli polynomials and q -sine hypergeometric Bernoulli polynomials given in (32) and (33).

Note that

$$\mathbb{B}_{j,q}^{(c)}(M, N, 0, \nu) := \mathbb{B}_{j,q}^{(c)}(M, N, \nu) \text{ and } \mathbb{B}_{j,q}^{(s)}(M, N, 0, \nu) := \mathbb{B}_{j,q}^{(s)}(M, N, \nu).$$

Theorem 9. Let $j \geq 1$. Then

$$D_{q,\psi} \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) = [j]_q \mathbb{B}_{j-1,q}^{(c)}(M, N, \psi, \nu), \quad (69)$$

$$D_{q,\nu} \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) = -[j]_q \mathbb{B}_{j-1,q}^{(s)}(M, N, \psi, q\nu), \quad (70)$$

and

$$D_{q,\psi} \mathbb{B}_{j,q}^{(s)}(M, N, \psi, \nu) = [j]_q \mathbb{B}_{j-1,q}^{(s)}(M, N, \psi, \nu), \quad (71)$$

$$D_{q,\nu} \mathbb{B}_{j,q}^{(s)}(M, N, \psi, \nu) = [j]_q \mathbb{B}_{j-1,q}^{(c)}(M, N, \psi, q\nu). \quad (72)$$

Proof. The proofs of Theorem 9 are similar to those of Theorem 6. So we omit them. \square

Theorem 10. For $j \geq 1$, we have

$$\mathbb{B}_{j,q}^{(c)}(M, N, (\psi \oplus w)_q, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j,q}^{(c)}(M, N, (\psi \oplus w)_q, \nu) w^k, \quad (73)$$

and

$$\mathbb{B}_{j,q}^{(s)}(M, N, (\psi \oplus w)_q, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j,q}^{(s)}(M, N, (\psi, \nu) w^k. \quad (74)$$

Proof. Using (67), we have

$$\begin{aligned} \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, (\psi \oplus w)_q, \nu) \frac{z^j}{[j]_q!} &= \frac{e_q((\psi \oplus w)_q z) \cos_q(\nu z)}{{}_2\phi_1 \left(\begin{matrix} q^{M+1}, 0 \\ q^{M+N+1} \end{matrix} \middle| q(1-q)z \right)} \\ &= \frac{e_q(\psi z) \cos_q(\nu z) e_q(wz)}{{}_2\phi_1 \left(\begin{matrix} q^{M+1}, 0 \\ q^{M+N+1} \end{matrix} \middle| q(1-q)z \right)} \\ &= \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, (\psi, \nu) \frac{z^j}{[j]_q!} \right) \left(\sum_{k=0}^{\infty} w^k \frac{z^k}{[k]_q!} \right) \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j,q}^{(c)}(M, N, (\psi, \nu) w^k \right) \frac{z^j}{[j]_q!}, \end{aligned} \quad (75)$$

which means (73). (74) can be derived similarly. So we omit it. \square

Now, we provide some corollaries of Theorem 10 as follows.

Corollary 4. Let $j \geq 0$. We have

$$\mathbb{B}_{j,q}^{(c)}(M, N, (\psi \oplus 1)_q, \nu) - \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) = \sum_{k=0}^{j-1} \binom{n}{k}_q \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu),$$

and

$$\mathbb{B}_{j,q}^{(s)}(M, N, (\psi \oplus 1)_q, \nu) - \mathbb{B}_{j,q}^{(s)}(M, N, \psi, \nu) = \sum_{k=0}^{j-1} \binom{n}{k}_q \mathbb{B}_{j,q}^{(s)}(M, N, \psi, \nu).$$

Corollary 5. *Let $j \geq 0$. Then*

$$\mathbb{B}_{j,q}^{(c)}(N, (\psi \oplus w)_q, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j,q}^{(c)}(N, (\psi, \nu)w^k,$$

and

$$\mathbb{B}_{j,q}^{(s)}(N, (\psi \oplus w)_q, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j,q}^{(s)}(N, (\psi, \nu)w^k.$$

Theorem 11. *Let $j \geq 1$. Then*

$$C_{j,q}(\psi, \nu) = \frac{\Gamma_q(M+N+1)}{\Gamma_q(M+1)} \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(c)}(M, N, \psi, \nu) \frac{\Gamma_q(M+k+1)}{\Gamma_q(M+N+k+1)}, \quad (76)$$

and

$$S_{j,q}(\psi, \nu) = \frac{\Gamma_q(M+N+1)}{\Gamma_q(M+1)} \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(s)}(M, N, \psi, \nu) \frac{\Gamma_q(M+k+1)}{\Gamma_q(M+N+k+1)}. \quad (77)$$

Proof. From (67), we have

$$\begin{aligned} e_q(\psi z) \cos_q(\nu z) &= {}_2\phi_1 \left(\begin{matrix} q^{M+1}, 0 \\ q^{M+N+1} \end{matrix} \middle| q(1-q)z \right) \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!} \right) \\ &= \left(\sum_{j=0}^{\infty} \frac{(q^{M+1}; q)_j}{(q^{M+N+1}; q)_j} \frac{z^j}{[j]_q!} \right) \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!} \right) \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \frac{\mathbb{B}_{j-k,q}^{(c)}(M, N, \psi, \nu)}{[j-k]_q!} \frac{(q^{M+1}; q)_k}{(q^{M+N+1}; q)_k} \frac{1}{[k]_q!} \right) z^j \\ &= \sum_{j=0}^{\infty} \left(\frac{\Gamma_q(M+N+1)}{\Gamma_q(M+1)} \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(c)}(M, N, \psi, \nu) \frac{\Gamma_q(M+k+1)}{\Gamma_q(M+N+k+1)} \right) \frac{z^j}{[j]_q!}. \end{aligned} \quad (78)$$

Therefore, by (67) and (78), we obtain the result (76). Similarly, we can readily acquire (77). So we omit the details. \square

We give the following results.

Corollary 6. *For real $M > 0$, $N > 0$ and $j \geq 0$. Then*

$$\sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(c)}(M, N, 0, 0) \frac{\Gamma_q(M+k+1)}{\Gamma_q(M+N+k+1)} = \begin{cases} \frac{\Gamma_q(M+1)}{\Gamma_q(M+N+1)}, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

and

$$\sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(c)}(M, N, 0, 0) \frac{\Gamma_q(M+k+1)}{\Gamma_q(M+N+k+1)} = 0.$$

Theorem 12. *For real $M > 0$, $N > 0$ and $j \geq 1$. Then*

$$\int_0^1 (1 \ominus q\psi)_q^{N-1} \psi^{M-1} \mathbb{B}_{k,q}^{(c)}(M, N, \psi, \nu) d_q \psi = \Gamma_q(N) \sum_{k=0}^j \binom{j}{k}_q B_q(N, M+j-k) \mathbb{B}_{k,q}^{(c)}(M, N, \nu), \quad (79)$$

and

$$\int_0^1 (1 \ominus q\psi)_q^{N-1} \psi^{M-1} \mathbb{B}_{k,q}^{(s)}(M, N, \psi, \nu) d_q \psi = \Gamma_q(N) \sum_{k=0}^j \binom{j}{k}_q B_q(N, M+j-k) \mathbb{B}_{k,q}^{(s)}(M, N, \nu). \quad (80)$$

Proof. From (16) and (67), we have

$$\begin{aligned} & \int_0^1 (1 \ominus q\psi)_q^{N-1} \psi^{M-1} \mathbb{B}_{k,q}^{(c)}(M, N, \psi, \nu) d_q \psi \\ &= \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{k,q}^{(c)}(M, N, \nu) \int_0^1 (1 \ominus q\psi)_q^{N-1} \psi^{M+j-k-1} d_q \psi \\ &= \sum_{k=0}^j \binom{j}{k}_q \frac{\Gamma_q(N) \Gamma_q(M+j-k)}{\Gamma_q(M+N+j-k)} \mathbb{B}_{k,q}^{(c)}(M, N, \nu) \\ &= \Gamma_q(N) \sum_{k=0}^j \binom{j}{k}_q \frac{\Gamma_q(N) \Gamma_q(M+j-k)}{\Gamma_q(M+N+j-k)} \mathbb{B}_{k,q}^{(c)}(M, N, \nu), \end{aligned} \quad (81)$$

which means the asserted result (79). The other can be done similarly. So we omit the proof. \square

Theorem 13. For real $M > 0$, $N > 0$ and a be a non-zero complex number. Then

$$\mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) = \sum_{k=0}^j \sum_{r=0}^k \binom{j}{k}_q \binom{k}{r}_q (-1)^r q^{\binom{r}{2}} \psi^k \mathbb{B}_{j-k,q}^{(c)}(M, N, a\psi, \nu), \quad (82)$$

and

$$\mathbb{B}_{j,q}^{(s)}(M, N, \psi, \nu) = \sum_{k=0}^j \sum_{r=0}^k \binom{j}{k}_q \binom{k}{r}_q (-1)^r q^{\binom{r}{2}} \psi^k \mathbb{B}_{j-k,q}^{(c)}(M, N, a\psi, \nu). \quad (83)$$

Proof. From (67), we have

$$\begin{aligned} & \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!} = \frac{e_q(\psi z) \cos_q(\nu z)}{{}_2\phi_1 \left(\begin{matrix} q^{M+1}, & 0 \\ q^{M+N+1} & | q(1-q)z \end{matrix} \right)} \\ &= \frac{e_q(a\psi z)}{{}_2\phi_1 \left(\begin{matrix} q^{M+1}, & 0 \\ q^{M+N+1} & | q(1-q)z \end{matrix} \right)} \frac{e_q(\psi z) \cos_q(\nu z)}{e_q(a\psi z)} \\ &= \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, a\psi, \nu) \frac{z^j}{[j]_q!} \right) e_q(\psi z) E_q(-a\psi z) \\ &= \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(c)}(M, N, a\psi, \nu) \frac{z^j}{[j]_q!} \right) \sum_{j=0}^{\infty} \sum_{r=0}^j \binom{j}{r}_q (-1)^r q^{\binom{r}{2}} \frac{\psi^j z^j}{[j]_q!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \sum_{r=0}^k \binom{j}{k}_q \binom{k}{r}_q (-1)^r q^{\binom{r}{2}} \psi^k \mathbb{B}_{j-k,q}^{(c)}(M, N, a\psi, \nu) \right) \frac{z^j}{[j]_q!}. \end{aligned} \quad (84)$$

\square

We now give the following Corollary.

Corollary 7. *For real $M > 0$, $N > 0$ and a be a non-zero complex number. Then*

$$\mathbb{B}_{j,q}^{(c)}(M, N, \frac{\psi}{a}, \nu) = \sum_{k=0}^j \sum_{r=0}^k \binom{j}{k}_q \binom{k}{r}_q (-1)^r q^{\binom{r}{2}} a^{-k} \psi^k \mathbb{B}_{j-k,q}^{(c)}(M, N, \psi, \nu), \quad (85)$$

and

$$\mathbb{B}_{j,q}^{(s)}(M, N, \frac{\psi}{a}, \nu) = \sum_{k=0}^j \sum_{r=0}^k \binom{j}{k}_q \binom{k}{r}_q (-1)^r q^{\binom{r}{2}} a^{-k} \psi^k \mathbb{B}_{j-k,q}^{(s)}(M, N, \psi, \nu). \quad (86)$$

Here, we consider the higher-order q -cosine and q -sine hypergeometric Bernoulli polynomials by

$$\frac{e_q(\psi z) \cos_q(\nu z)}{\left[{}_2\phi_1 \left(\begin{matrix} q^{M+1}, 0 \\ q^{M+N+1} \end{matrix} \middle| q(1-q)z \right) \right]^\tau} = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau,c)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!}, \quad (87)$$

and

$$\frac{e_q(\psi z) \sin_q(\nu z)}{\left[{}_2\phi_1 \left(\begin{matrix} q^{M+1}, 0 \\ q^{M+N+1} \end{matrix} \middle| q(1-q)t \right) \right]^\tau} = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau,s)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!}, \quad (88)$$

for $\tau \in \mathbb{C}$ and $M, N > 0$.

We note that

$$\mathbb{B}_{j,q}^{(\tau,c)}(M, N, 0, 0) = \mathbb{B}_{j,q}^{(\tau)}(M, N) \text{ and } \mathbb{B}_{j,q}^{(\tau,s)}(M, N, 0, 0) = 0,$$

where $\mathbb{B}_{j,q}^{(\tau)}(M, N)$ are called the higher-order q -hypergeometric Bernoulli numbers.

Remark 2. *The higher-order q -cosine and q -sine hypergeometric Bernoulli polynomials are given by*

$$\frac{\cos_q(\nu z)}{\left[{}_2\phi_1 \left(\begin{matrix} q^{M+1}, 0 \\ q^{M+N+1} \end{matrix} \middle| q(1-q)z \right) \right]^\tau} = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau,c)}(M, N, \nu) \frac{z^j}{[j]_q!},$$

and

$$\frac{\sin_q(\nu z)}{\left[{}_2\phi_1 \left(\begin{matrix} q^{M+1}, 0 \\ q^{M+N+1} \end{matrix} \middle| q(1-q)t \right) \right]^\tau} = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau,s)}(M, N, \nu) \frac{z^j}{[j]_q!},$$

for $\tau \in \mathbb{C}$ and $M, N > 0$.

Theorem 14. *For $j \geq 0$. Then*

$$\mathbb{B}_{j,q}^{(\tau,c)}(M, N, \psi, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,c)}(M, N, \nu) \psi^k, \quad (89)$$

and

$$\mathbb{B}_{j,q}^{(\tau,s)}(M, N, \psi, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,s)}(M, N, \nu) \psi^k. \quad (90)$$

Proof. From (87), it follows that

$$\begin{aligned} \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau,c)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!} &= \frac{e_q(\psi z) \cos_q(\nu z)}{\left[{}_2\phi_1 \left(\begin{matrix} q^{M+1}, & 0 \\ & q^{M+N+1} \end{matrix} \middle| q(1-q)z \right) \right]^{\tau}} \\ &= \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau,c)}(M, N, \nu) \frac{z^j}{[j]_q!} \right) \left(\sum_{k=0}^{\infty} \psi^k \frac{z^k}{[k]_q!} \right) \end{aligned} \quad (91)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,c)}(M, N, \nu) \psi^k \right) \frac{z^j}{[j]_q!}. \quad (92)$$

(91) means the asserted result (89). (90) can be done similarly using (88). So we omit the proof. \square

Theorem 15. For $j \geq 0$. Then

$$\mathbb{B}_{j,q}^{(\tau+\beta,c)}(M, N, (\psi \oplus w)_q, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,c)}(M, N, \psi, \nu) \mathbb{B}_{k,q}^{(\beta,c)}(M, N, w), \quad (93)$$

and

$$\mathbb{B}_{j,q}^{(\tau+\beta,s)}(M, N, (\psi \oplus w)_q, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,s)}(M, N, \psi, \nu) \mathbb{B}_{k,q}^{(\beta,s)}(M, N, w). \quad (94)$$

Proof. Replacing τ by $\tau + \beta$ and ψ by $(\psi \oplus w)_q$ in (87), we have

$$\begin{aligned} \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau+\beta,c)}(M, N, \psi + w, \nu) \frac{z^j}{[j]_q!} &= \frac{e_q((\psi \oplus w)_q z) \cos_q(\nu z)}{\left[{}_2\phi_1 \left(\begin{matrix} q^{M+1}, & 0 \\ & q^{M+N+1} \end{matrix} \middle| q(1-q)z \right) \right]^{\tau+\beta}} \\ &= \left(\sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau,c)}(M, N, \psi, \nu) \frac{z^j}{[j]_q!} \right) \left(\sum_{k=0}^{\infty} \mathbb{B}_{k,q}^{(\beta,c)}(M, N, w) \frac{z^k}{[k]_q!} \right) \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,c)}(M, N, \psi, \nu) \mathbb{B}_{k,q}^{(\beta,c)}(M, N, w) \right) \frac{z^j}{[j]_q!}. \end{aligned} \quad (95)$$

(94) means the asserted result (92). (93) can be done similarly using (88). So we omit the proof. \square

Corollary 8. For $j \geq 0$. Then

$$C_j(\psi, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,c)}(M, N, \nu) \mathbb{B}_{k,q}^{(-\tau)}(M, N, \psi), \quad (96)$$

and

$$S_j(\psi, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,s)}(M, N, \nu) \mathbb{B}_{k,q}^{(-\tau)}(M, N, \psi), \quad (97)$$

where the higher-order hypergeometric q -Bernoulli polynomials are provided by (see Ryoo et al., 2020)

$$\frac{e_q(\psi z)}{\left[{}_2\phi_1 \left(\begin{matrix} q^{M+1}, & 0 \\ & q(1-q)z \end{matrix} \middle| q \right) \right]^\tau} = \sum_{j=0}^{\infty} \mathbb{B}_{j,q}^{(\tau)}(M, N, \psi) \frac{z^j}{[j]_q!}.$$

Corollary 9. For $j \geq 0$. Then

$$\mathbb{B}_{j,q}^{(\tau+\beta,c)}(M, N, \psi, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,c)}(M, N, \psi, \nu) \mathbb{B}_{k,q}^{(\beta,c)}(M, N), \tag{98}$$

and

$$\mathbb{B}_{j,q}^{(\tau+\beta,s)}(M, N, \psi, \nu) = \sum_{k=0}^j \binom{j}{k}_q \mathbb{B}_{j-k,q}^{(\tau,s)}(M, N, \psi, \nu) \mathbb{B}_{k,q}^{(\beta,s)}(M, N). \tag{99}$$

5 Computational Values and Graphical Representations of the q -Cosine and q -Sine Hypergeometric Bernoulli Polynomials

In this section, certain zeros of q -cosine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{j,q}^{(c)}(N, \psi, \nu)$ and beautifully graphical representations are shown.

Let $N = 5$. The first few q -cosine hypergeometric Bernoulli polynomials are as follows:

$$\begin{aligned} \mathbb{B}_{0,q}^{(c)}(N, \psi, \nu) &= 1, \\ \mathbb{B}_{1,q}^{(c)}(N, \psi, \nu) &= \psi - \frac{[5]_q!}{[6]_q!}, \\ \mathbb{B}_{2,q}^{(c)}(N, \psi, \nu) &= -q\nu^2 + \psi^2 + \frac{[2]_q!([5]_q!)^2}{([6]_q!)^2} - \frac{\psi[2]_q![5]_q}{[6]_q!} - \frac{[2]_q![5]_q}{[7]_q!}, \\ \mathbb{B}_{3,q}^{(c)}(N, \psi, \nu) &= \psi^3 - \frac{q\nu^2\psi[3]_q!}{[2]_q!} - \frac{[3]_q!([5]_q!)^3}{([6]_q!)^3} + \frac{\psi[3]_q!([5]_q!)^2}{([6]_q!)^2} + \frac{q\nu^2[3]_q![5]_q!}{[2]_q![6]_q!} \\ &\quad - \frac{\psi^2[3]_q![5]_q!}{[2]_q![6]_q!} - \frac{\psi[3]_q![5]_q!}{[7]_q!} + \frac{2[3]_q!([5]_q!)^2}{[6]_q![7]_q!} - \frac{[3]_q![5]_q!}{[8]_q!}, \\ \mathbb{B}_{4,q}^{(c)}(N, \psi, \nu) &= q^6\nu^4 + \psi^4 - \frac{q\nu^2\psi^2[4]_q!}{([2]_q!)^2} + \frac{[4]_q!([5]_q!)^4}{([6]_q!)^4} - \frac{\psi[4]_q!([5]_q!)^3}{([6]_q!)^3} \\ &\quad - \frac{q\nu^2[4]_q!([5]_q!)^2}{[2]_q!([6]_q!)^2} + \frac{\psi^2[4]_q!([5]_q!)^2}{[2]_q!([6]_q!)^2} + \frac{q\nu^2\psi[4]_q![5]_q!}{[2]_q![6]_q!} \\ &\quad - \frac{\psi^3[4]_q![5]_q!}{[3]_q![6]_q!} + \frac{[4]_q!([5]_q!)^2}{([7]_q!)^2} + \frac{q\nu^2[4]_q![5]_q!}{[2]_q![7]_q!} - \frac{\psi^2[4]_q![5]_q!}{[2]_q![7]_q!} \\ &\quad - \frac{3[4]_q!([5]_q!)^3}{([6]_q!)^2[7]_q!} + \frac{2\psi[4]_q!([5]_q!)^2}{[6]_q![7]_q!} - \frac{\psi[4]_q![5]_q!}{[8]_q!} + \frac{2[4]_q!([5]_q!)^2}{[6]_q![8]_q} - \frac{[4]_q![5]_q!}{[9]_q!}. \end{aligned}$$

We derive the zeros of the q -cosine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(c)}(N, \psi, \nu)$. We plot the beautiful zeros of the q -cosine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(c)}(N, \psi, \nu) = 0$ for $n = 30$ (Figure 1).

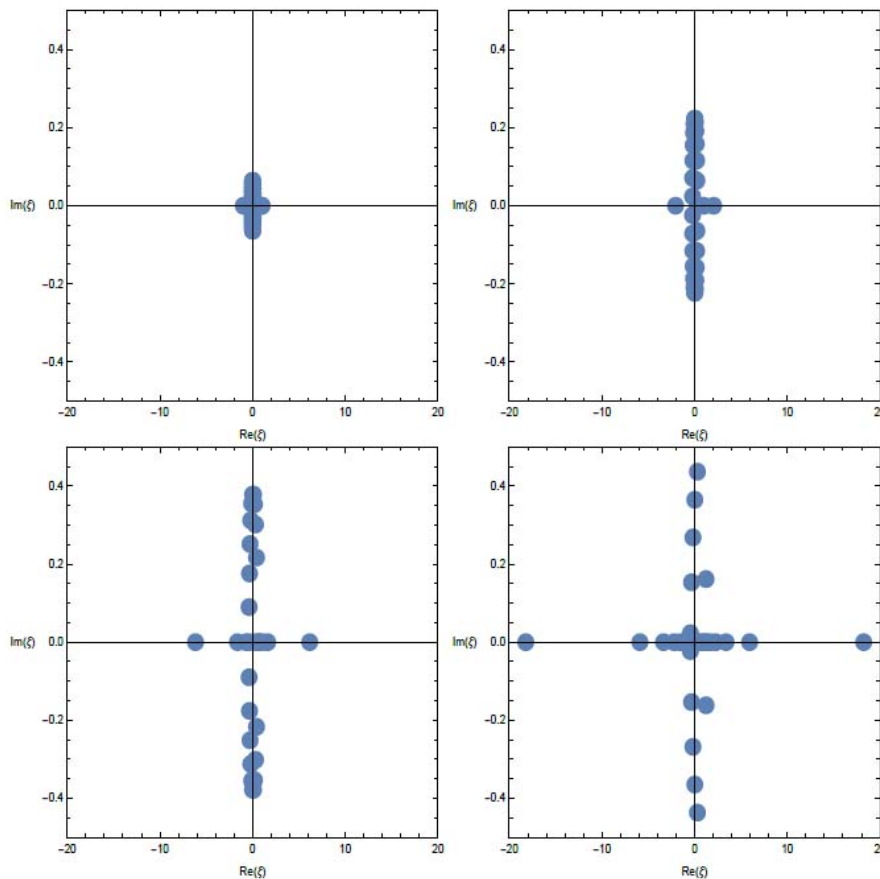


Figure 1: Zeros of $\mathbb{B}_{n,q}^{(c)}(\psi, \nu; N)$

In Figure 1 (top-left), we choose $\nu = 3, N = 5$, and $q = \frac{1}{10}$. In Figure 1 (top-right), we choose $\nu = 3, N = 5$, and $q = \frac{3}{10}$. In Figure 1 (bottom-left), we choose $\nu = 3, N = 5$, and $q = \frac{7}{10}$. In Figure 1 (bottom-right), we choose $\nu = 3, N = 5$, and $q = \frac{9}{10}$.

Stacks of zeros of the q -cosine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(c)}(N, \psi, \nu) = 0$ for $1 \leq n \leq 30$, forming a 3D structure, are presented (Figure 2).

In Figure 2 (top-left), we plot stacks of zeros of the q -cosine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(c)}(N, \psi, \nu) = 0$ for $1 \leq n \leq 30$, $q = \frac{9}{10}, \nu = 3, N = 5$. In Figure 2(top-right), we draw x and y axes but no z axis in three dimensions. In Figure 2 (bottom-left), we draw y and z axes but no x axis in three dimensions. In Figure 2(bottom-right), we draw x and z axes but no y axis in three dimensions.

We plot the real zeros of the q -cosine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(c)}(N, \psi, \nu) = 0$ for $1 \leq n \leq 30$ (Figure 3).

In Figure 3(top-left), we choose $\nu = 3, N = 5$, and $q = \frac{1}{10}$. In Figure 3(top-right), we choose $\nu = 3, N = 5$, and $q = \frac{3}{10}$. In Figure 3(bottom-left), we choose $\nu = 3, N = 5$, and $q = \frac{7}{10}$. In Figure 3(bottom-right), we choose $\nu = 3, N = 5$, and $q = \frac{9}{10}$.

Next, we computed an approximate solution satisfying the q -cosine hypergeometric Bernoulli polynomials $\mathbb{B}_{n,q}^{(c)}(N, \psi, \nu) = 0$ for $q = \frac{9}{10}$. The results are provided in Table 1.

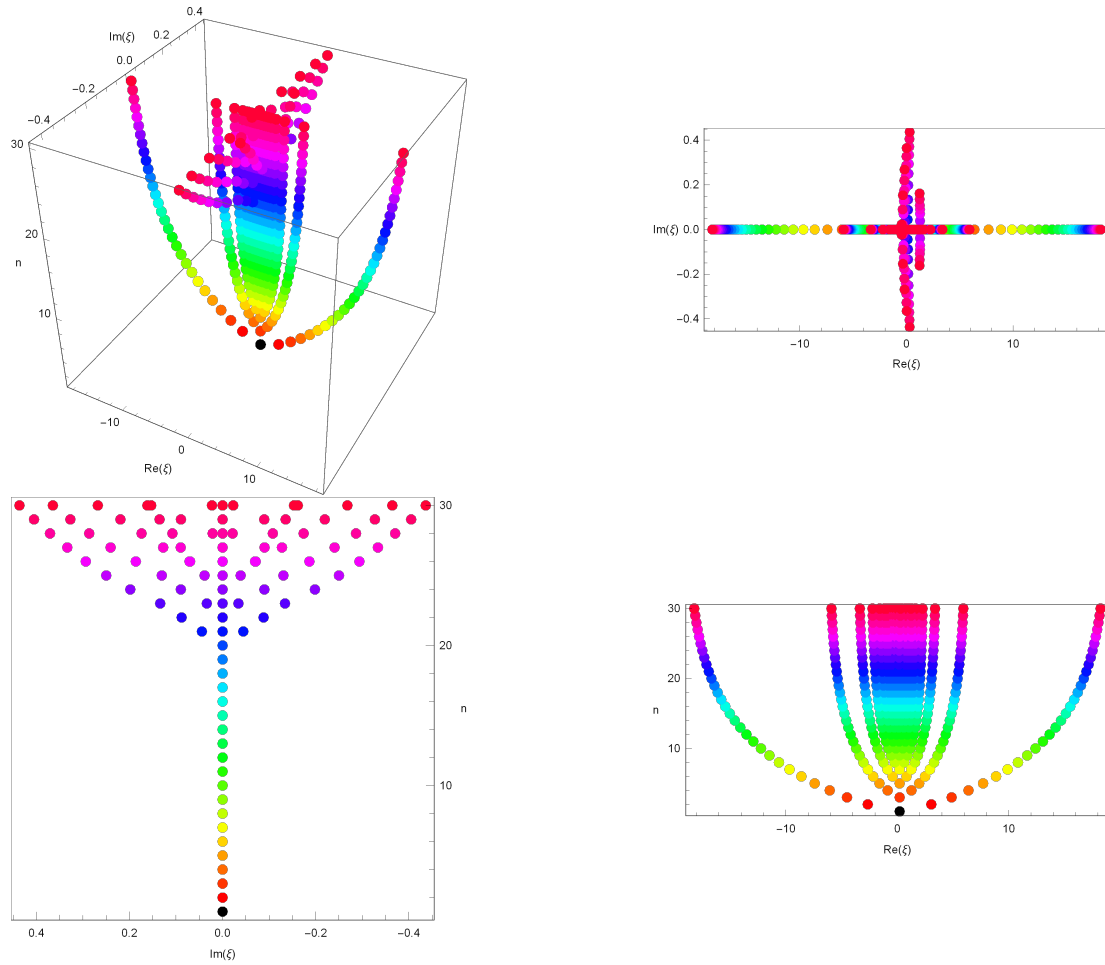


Figure 2: Zeros of $\mathbb{B}_{n,q}^{(c)}(\psi, \nu; N)$

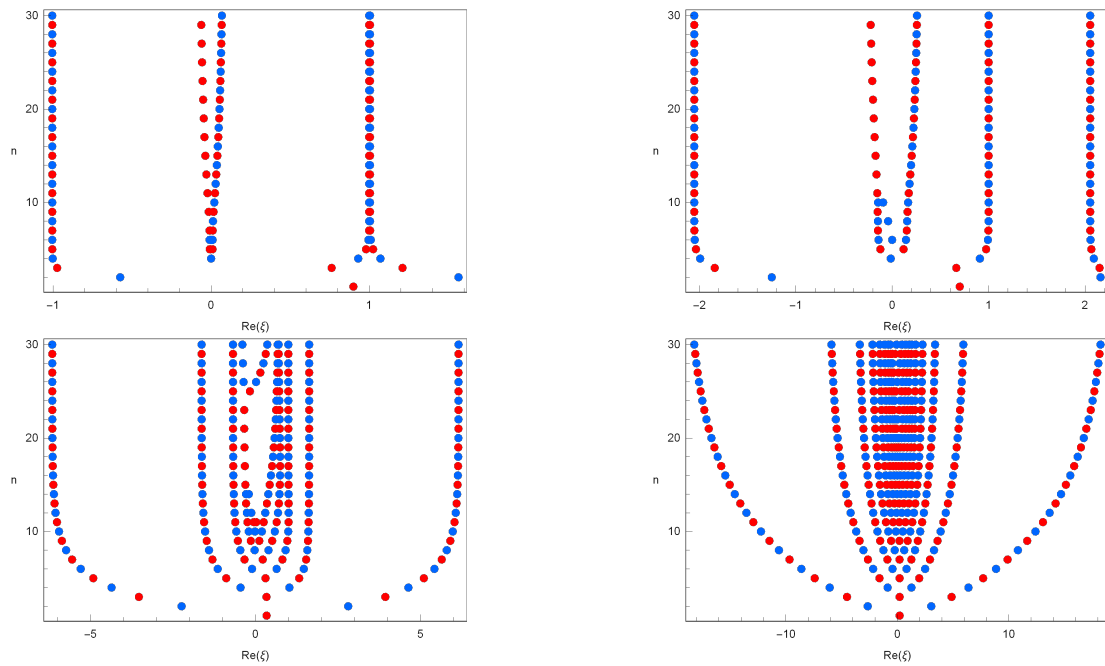


Figure 3: Real zeros of $\mathbb{B}_{n,q}^{(c)}(\psi, \nu; N)$

Table 1. Approximate solutions of $\mathbb{B}_{n,q}^{(c)}(\psi, 3; 5) = 0, \psi \in \mathbb{R}$

degree n	ψ
1	0.21342
2	-2.6490, 3.0545
3	-4.5118, 0.21303, 4.8772
4	-6.0611, -0.86614, 1.2732, 6.3880
5	-7.4100, -1.5976, 0.21055, 1.9685, 7.7026
6	-8.6032, -2.1672, -0.42592, 0.83160, 2.4995 8.8652
7	-9.6660, -2.6371, -0.88526, 0.20480, 1.2630, 2.9330, 9.9010
8	-10.616, -3.0364, -1.2428, -0.22834, 0.62393, 1.5888, 3.2994, 10.827
9	-11.467, -3.3817, -1.5331, -0.55397, 0.19452, 0.92985, 1.8469, 3.6156, 11.657
10	-12.231, -3.6837, -1.7752, -0.80940, -0.12363, 0.49819, 1.1642, 2.0579, 3.8917, 12.401

Here, certain zeros of q -sine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{j,q}^{(s)}(N, \psi, \nu)$ and beautifully graphical representations are shown.

The first few q -sine hypergeometric Bernoulli polynomials are as follows:

$$\begin{aligned} \mathbb{B}_{0,q}^{(s)}(N, \psi, \nu) &= 0, \\ \mathbb{B}_{1,q}^{(s)}(N, \psi, \nu) &= \nu, \\ \mathbb{B}_{2,q}^{(s)}(N, \psi, \nu) &= \nu\psi[2]_q! - \frac{\nu[2]_q![5]_q!}{[6]_q!}, \\ \mathbb{B}_{3,q}^{(s)}(N, \psi, \nu) &= -q^3\nu^3 + \frac{\nu\psi^2[3]_q!}{[2]_q!} + \frac{\nu[3]_q!([5]_q!)^2}{([6]_q!)^2} - \frac{\nu\psi[3]_q![5]_q!}{[6]_q!} - \frac{\nu[3]_q![5]_q!}{[7]_q!}, \\ \mathbb{B}_{4,q}^{(s)}(N, \psi, \nu) &= -\frac{q^3\nu^3\psi[4]_q!}{[3]_q!} + \frac{\nu\psi^3[4]_q!}{[3]_q!} - \frac{\nu[4]_q!([5]_q!)^3}{([6]_q!)^3} + \frac{\nu\psi[4]_q!([5]_q!)^2}{([6]_q!)^2} \\ &+ \frac{2\nu[4]_q!([5]_q!)^2}{[6]_q![7]_q!} - \frac{\nu[4]_q![5]_q!}{[8]_q!} \\ &+ \frac{2\nu[4]_q!([5]_q!)^2}{[6]_q![7]_q!} - \frac{\nu[4]_q![5]_q!}{[8]_q!}, \\ \mathbb{B}_{5,q}^{(s)}(N, \psi, \nu) &= q^{10}\nu^5 - \frac{q^3\nu^3\psi^2[5]_q!}{[2]_q![3]_q!} + \frac{\nu\psi^4[5]_q!}{[4]_q!} + \frac{\nu([5]_q!)^5}{([6]_q!)^4} - \frac{\nu\psi([5]_q!)^4}{([6]_q!)^3} \\ &+ \frac{\nu\psi^2([5]_q!)^3}{[2]_q!([6]_q!)^2} - \frac{q^3\nu^3([5]_q!)^3}{[3]_q!([6]_q!)^2} + \frac{q^3\nu^3\psi([5]_q!)^2}{[3]_q![6]_q!} - \frac{\nu\psi^3([5]_q!)^2}{[3]_q![6]_q!} \\ &+ \frac{\nu([5]_q!)^3}{([7]_q!)^2} - \frac{\nu\psi^2([5]_q!)^2}{[2]_q![7]_q!} + \frac{q^3\nu^3([5]_q!)^2}{[3]_q![7]_q!} - \frac{3\nu([5]_q!)^4}{([6]_q!)^2[7]_q!} \\ &+ \frac{2\nu\psi([5]_q!)^3}{[6]_q![7]_q!} - \frac{\nu\psi([5]_q!)^2}{[8]_q!} + \frac{2\nu([5]_q!)^3}{[6]_q![8]_q!} - \frac{\nu([5]_q!)^2}{[9]_q!}. \end{aligned}$$

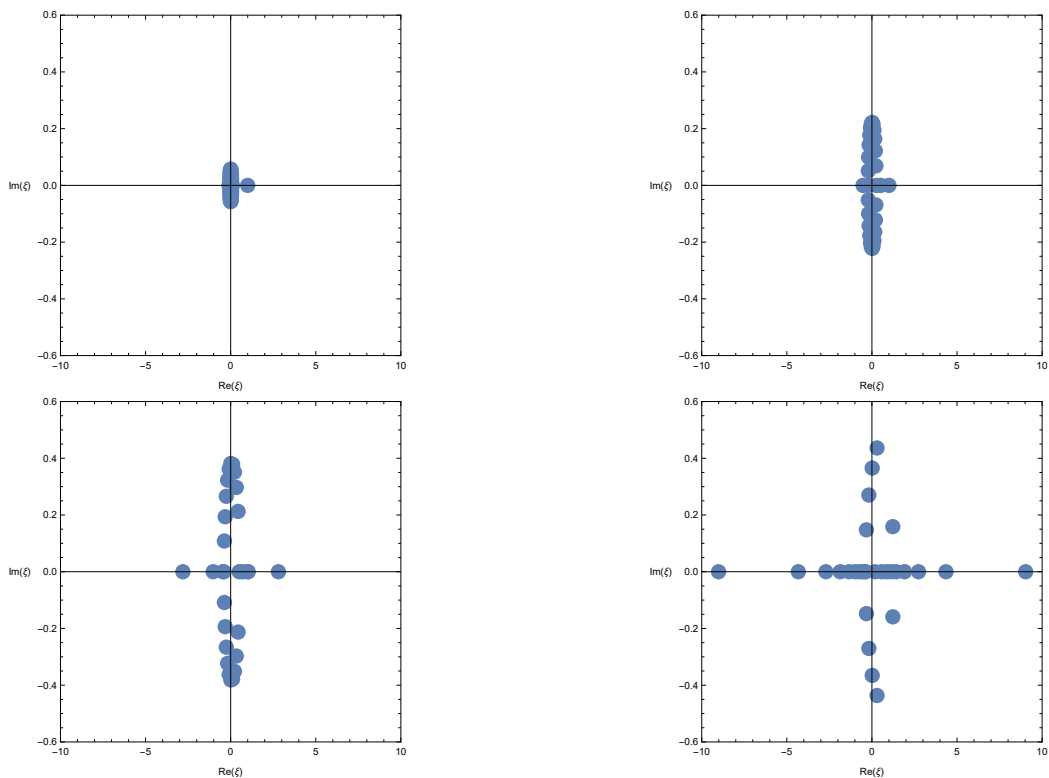


Figure 4: Zeros of $\mathbb{B}_{n,q}^{(s)}(\psi, \nu; N)$

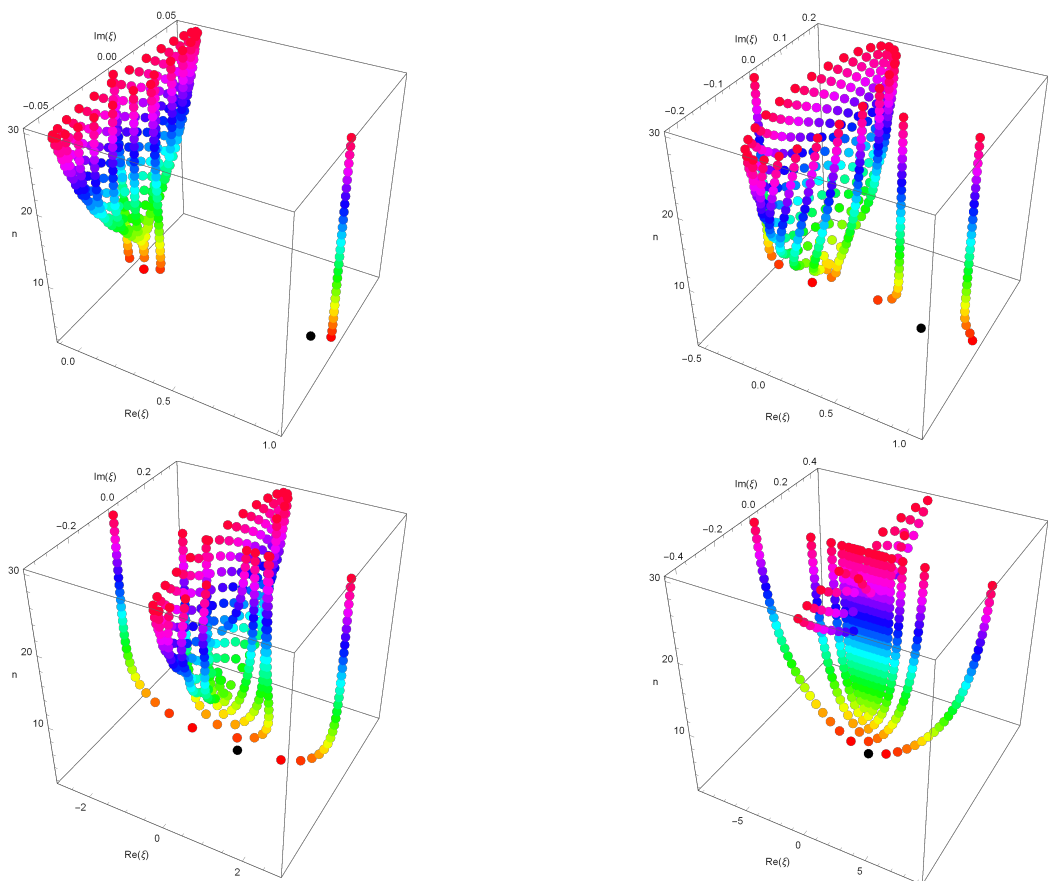


Figure 5: Zeros of $\mathbb{B}_{n,q}^{(s)}(\psi, \nu; N)$

We investigate the beautiful zeros of the q -sine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(s)}(N, \psi, \nu)$ by using a computer. We plot the zeros of the q -sine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(s)}(N, \psi, \nu) = 0$ for $n = 30$ (Figure 4).

In Figure 4 (top-left), we choose $\nu = 3, N = 5$, and $q = \frac{1}{10}$. In Figure 4 (top-right), we choose $\nu = 3, N = 5$, and $q = \frac{3}{10}$. In Figure 4 (bottom-left), we choose $\nu = 3, N = 5$, and $q = \frac{7}{10}$. In Figure 4 (bottom-right), we choose $\nu = 3, N = 5$, and $q = \frac{9}{10}$.

Stacks of zeros of the q -sine hypergeometric Bernoulli polynomials of complex variable $\mathbb{B}_{n,q}^{(s)}(N, \psi, \nu) = 0$ for $1 \leq n \leq 30$, forming a 3D structure, are presented (Figure 5).

In Figure 5 (top-left), we choose $\nu = 3, N = 5$, and $q = \frac{1}{10}$. In Figure 5 (top-right), we choose $\nu = 3, N = 5$, and $q = \frac{3}{10}$. In Figure 5 (bottom-left), we choose $\nu = 3, N = 5$, and $q = \frac{7}{10}$. In Figure 5 (bottom-right), we choose $\nu = 3, N = 5$, and $q = \frac{9}{10}$.

Next, we computed an approximate solution satisfying the q -sine hypergeometric Bernoulli polynomials $\mathbb{B}_{n,q}^{(s)}(N, \psi, \nu) = 0$ for $q = \frac{9}{10}$. The results are given in Table 1.

Table 1. Approximate solutions of $\mathbb{B}_{n,q}^{(s)}(\psi, 3; 5) = 0, \psi \in \mathbb{R}$

degree n	ψ
2	0.21342
3	-1.3636, 1.7691
4	-2.3953, 0.21212, 2.7616
5	-3.2122, -0.59575, 1.0031, 3.5389
6	-3.8996, -1.1624, 0.20817, 1.5375, 4.1902
7	-4.4937, -1.6023, -0.31051, 0.71239, 1.9417, 4.7525
8	-5.0146, -1.9614, -0.69352, 0.20030, 1.0717, 2.2657, 5.2453
9	-5.4749, -2.2633, -0.99284, -0.16822 0.55478, 1.3443, 2.5348, 5.6809
10	-5.8838, -2.5217, -1.2351, -0.44901, 0.18734, 0.81950, 1.5588, 2.7634, 6.0679
11	-6.2482, -2.7457, -1.4363, -0.66987, -0.090513, 0.44990, 1.0251, 1.7319, 2.9607, 6.4129

6 Conclusions

In this study, we have introduced two new extensions of the Bernoulli polynomials and numbers by utilizing q -cosine, q -sine, q -hypergeometric and q -exponential function, which we called q -cosine and q -sine hypergeometric Bernoulli polynomials. Afterthat, we have derived several properties and formulas for these polynomials, such as summation formulas, addition formulas, q -derivative properties, representations by definite q -integrals and some correlations. Also, we have considered q -cosine and q -sine hypergeometric Bernoulli polynomials with two parameters and given some relations and identities. Finally, we have examined some computational values, given by tables, and the beautiful zeros representations, given by figures, of the q -cosine hypergeometric Bernoulli polynomials and q -sine hypergeometric Bernoulli polynomials.

It can be considered that not only the idea of the present paper can be used for similar polynomials, but also the foregoing polynomials possess possible utilizations and applications in other scientific fields other than the applications at the end of the paper. In addition, advancing

the purpose of this article, we will proceed with this idea in our next research studies in several directions.

References

- Alam, N., Khan, W.A. & Ryoo, C.S. (2022). A note on Bell-based Apostol-type Frobenius-Euler polynomials of complex variable with its certain applications, *Mathematics*, 10(12), 2109.
- Al-Salam, W.A. (1959). q -Bernoulli numbers and polynomials, *Mathematische Nachrichten*, 17, 239-260.
- Al-Salam, W.A. (1976). q -Appell polynomials, *Annali di Matematica Pura ed Applicata*, 77(4), 31-45.
- Aoki, M., Komatsu, T. & Panda, G.K. (2019). Several properties of hypergeometric Bernoulli numbers, *Journal of Inequalities and Applications*, 2019(2019), 113.
- Avram, F. & Taqqu, M.S. (1987). Noncentral limit theorems and Appell polynomials, *Annals of Probability*, 15, 767-775.
- Corcino, C., Corcino, R. & Casquejo, J. (2022). Fourier Series Expansion and Integral Representation of Apostol-Type Frobenius–Euler Polynomials of Complex Parameters and Order α , *Symmetry*, 14(9), 1860.
- Guan, H. Khan, W.A. & Kizilates, C. (2023). On generalized bivariate (p, q) -Bernoulli-Fibonacci and generalized bivariate (p, q) -Bernoulli-Lucas polynomials, *Symmetry*, 15(4), 943.
- Howard, F.T. (1967). A sequence of numbers related to the exponential function, *Duke Mathematical Journal*, 34(3), 599-615.
- Howard, F.T. (1967). Some sequences of rational numbers related to the exponential function, *Duke Mathematical Journal*, 34(4), 701-716.
- Hu, S. & Kim, M.S. (2018). On hypergeometric Bernoulli numbers and polynomials, *Acta Mathematica Hungarica*, 154, 134-146.
- Hassen, A. & Nguyen, H.D. (2008). Noncentral limit theorems and Appell polynomials, *Annals of Probability*, 4(5), 767-774.
- Kac, V. & Cheung P. (2002). *Quantum calculus*. Springer New York, NY.
- Kang, J.Y. & Ryoo, C.S.(2020). Various structures of the roots and explicit properties of q -cosine Bernoulli polynomials and q -sine Bernoulli polynomials, *Mathematics*, 8(4), 463.
- Kamano, K. (2010). Sums of products of hypergeometric Bernoulli numbers, *Journal of Number Theory*, 130(10), 2259-2271.
- Khan, W.A. (2022). On generalized Lagrange-based Apostol type and related polynomials, *Kragujevac Journal of Mathematics*, 46(6), 865-882.
- Khan, W.A. (2022). A new class of higher-order hypergeometric Bernoulli polynomials associated with Hermite polynomials, *Boletim da Sociedade Paranaense de Matematica*, 2022(40), 1-14.
- Khan, W.A., Muhiuddin, G., Duran, U. & Al-Kadi, D. (2022). On (p, q) -Sine and (p, q) -Cosine Fubini polynomials, *Symmetry*, 14(3), 527.
- Khan, W.A., Alatawi, M.S., Ryoo, C.S. & Duran, U. (2023). Novel Properties of q -Sine-Based and q -Cosine-Based q -Fubini Polynomials, *Symmetry*, 15(2), 356.

- Khan, W.A., Alatawi, M.S. & Duran, U. (2023). Applications and Properties for Bivariate Bell-Based Frobenius-Type Eulerian Polynomials, *Journal of Function Spaces*, 2023, 5205867.
- Koekoek, R., Lesky, P.A. & Swarttouw, R.F. (2010). *Hypergeometric Orthogonal Polynomials and their q -Analogues*. Springer Berlin, Heidelberg.
- Mboutngam, S. & Sadjang, P.N. (2021). On q -hypergeometric Bernoulli polynomials and numbers, *Hacettepe Journal of Mathematics & Statistics*, 50(5), 1251-1267.
- Muhiuddin, G., Khan, W.A., Duran, U. & Al-Kadi, D. (2021). A new class of higher-order hypergeometric Bernoulli polynomials associated with Lagrange-Hermite polynomials, *Symmetry*, 13(4), 648.
- Nguyen, H.D. & Cheong, L.G. (2014). New convolution identities for hypergeometric Bernoulli polynomials, *Journal of Number Theory*, 137, 201-221.
- Norlund N.E. (1924). *Vorlesungen über Differenzenrechnung*. Springer-Verlag Berlin Heidelberg.
- Rao, Y., Khan, W.A., Araci, S. & Ryoo, C.S. (2023). Explicit Properties of Apostol-Type Frobenius-Euler Polynomials Involving q -Trigonometric Functions with Applications in Computer Modeling, *Mathematics*, 11(10), 2386.
- Ryoo, C.S. & Kang, J.Y. (2020). Explicit properties of q -cosine and q -sine Euler polynomials containing symmetric structures, *Symmetry*, 12(8), 1247.